

THE QUADRATIC WDVV SOLUTION $E_8(a_1)$

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ABSTRACT. We calculate explicitly the quadratic solution to the WDVV equations corresponds to the quasi-Coxeter conjugacy class $E_8(a_1)$ using the associated classical W -algebra.

A Frobenius algebra is a commutative associative algebra with unity e and an invariant non-degenerate bilinear form (\cdot, \cdot) . A **Frobenius manifold** is a manifold M with a smooth structure of Frobenius algebra on the tangent space $T_t M$ at any point $t \in M$ with certain compatibility conditions [7]. Globally, we require the metric (\cdot, \cdot) to be flat and the unity vector field e is constant with respect to it. In the flat coordinates (t^1, \dots, t^r) where $e = \frac{\partial}{\partial t^{r-1}}$ the compatibility conditions implies that there exist a function $\mathbb{F}(t^1, \dots, t^r)$ such that

$$\eta_{ij} = (\partial_{t^i}, \partial_{t^j}) = \partial_{t^{r-1}} \partial_{t^i} \partial_{t^j} \mathbb{F}(t)$$

and the structure constants of the Frobenius algebra is given by

$$C_{ij}^k = \eta^{kp} \partial_{t^p} \partial_{t^i} \partial_{t^j} \mathbb{F}(t)$$

where η^{ij} denote the inverse of the matrix η_{ij} . In this work, we consider Frobenius manifolds where the quasihomogeneity condition takes the form

$$(0.1) \quad \sum_{i=1}^r d_i t^i \partial_{t^i} \mathbb{F}(t) = (3-d) \mathbb{F}(t); \quad d_{r-1} = 1.$$

This condition defines **the degrees** d_i and **the charge** d of the Frobenius structure. If $\mathbb{F}(t)$ is an algebraic function we call M an **algebraic Frobenius manifold**. The associativity of Frobenius algebra implies the potential $\mathbb{F}(t)$ satisfies a system of partial differential equations known as **the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations**:

$$(0.2) \quad \partial_{t^i} \partial_{t^j} \partial_{t^k} \mathbb{F}(t) \eta^{kp} \partial_{t^p} \partial_{t^q} \partial_{t^n} \mathbb{F}(t) = \partial_{t^n} \partial_{t^j} \partial_{t^k} \mathbb{F}(t) \eta^{kp} \partial_{t^p} \partial_{t^q} \partial_{t^i} \mathbb{F}(t).$$

In topological field theory a solution to WDVV equations describes the a module space of two dimensional topological field theory [5].

Dubrovin conjecture on classification of algebraic Frobenius manifolds and hence algebraic WDVV solutions, is stated as follows: semisimple irreducible algebraic Frobenius manifolds with positive degrees d_i correspond to quasi-Coxeter (primitive) conjugacy classes in irreducible Coxeter groups. A quasi-Coxeter conjugacy class in an irreducible Coxeter group is a Conjugacy class which has no representative in a proper Coxeter subgroup.

There are two major results support the conjecture. First, the conjecture arises from studying the algebraic solutions to associated equations of isomonodromic deformation of algebraic Frobenius manifolds [7],[8]. It leads to quasi-Coxeter conjugacy classes in Coxeter groups by considering the classification of finite orbits of the braid group action on tuple of reflections obtained in [12]. Therefore, it remains the problem of constructing all these algebraic Frobenius manifolds. Second, Dubrovin constructed polynomial Frobenius structures on the orbit spaces of Coxeter groups [6]. Then Hertling [9] proved that these are all possible **polynomial Frobenius manifolds**. The isomonodromic deformation of Polynomial Frobenius manifolds lead to Coxeter conjugacy classes [7].

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The classification of polynomial Frobenius manifolds reveals a relation between the order and eigenvalues of the conjugacy class, and the charge and degrees of the corresponding Frobenius manifold. More precisely, If the order of the conjugacy class is $\kappa + 1$ and the eigenvalues are $\exp \frac{2\eta_i \pi i}{\kappa+1}$ then the charge of the Frobenius manifold is $\frac{\kappa-1}{\kappa+1}$ and the degrees are $\frac{\eta_i+1}{\kappa+1}$. We depend on this **weak relation** in considering a new examples of algebraic Frobenius manifolds. In [2] we continue the work of [10] and we began to develop a construction of algebraic Frobenius manifolds using classical W -algebras. This means we restrict ourself to conjugacy classes in Weyl groups. The examples obtained correspond, in the notations of [1], to the conjugacy classes $D_4(a_1)$ and $F_4(a_2)$.

In [3] we uniform the construction of polynomial Frobenius manifolds from classical W -algebras associated to regular nilpotent orbits. In [4] we extend this result and we uniform the construction of algebraic Frobenius manifolds form the classical W -algebras associated to subregular nilpotent orbits in the Lie algebra of type D_r where r is even and E_r . These subregular nilpotent orbits correspond to the regular quasi-Coxeter conjugacy classes $D_r(a_1)$ where r is even and $E_r(a_1)$, respectively.

In this work we write explicitly the potential of the algebraic Frobenius manifold $E_8(a_1)$. One of the main obstacles was that the minimal faithful matrix representation of the Lie algebra E_8 is the adjoint representation. This means that the elements of the Lie algebra are represented by square matrices of dimension 248. We overcome this problem by constructing instead the Weyl-Chevalley normal form. Another problem arises was the following. The Frobenius structure lives in a hypersurface which obtained by calculating the restriction of the invariant polynomials of the adjoint action to Slodowy slice. We avoid this calculation by returning to the method we used in [2] to obtain the algebraic Frobenius manifold $F_4(a_2)$. Which depends on the existence of a local Poisson structure compatible with the classical W -algebras (bihamiltonian structure).

In this paper we will not review the rich and deep theory behind the construction of the WDVV solution from classical W -algebras associated to nilpotent orbits. The interested reader may consult the paper [4] and [2] for details.

The resulting Frobenius manifold has the following potential $\mathbb{F}(t_1, t_2, \dots, t_8, Z)$ which has 303 monomial. Here Z is a solution of a quadratic equation where the coefficients are polynomials in t_1, \dots, t_6 and t_8 . We could not simplify the potential \mathbb{F} further. Any simplification comes with the price of losing the flat coordinates or having very large quadratic equation for Z . The quasihomogeneity reads

$$(0.3) \quad \left(\frac{1}{12}t_1\partial_{t_1} + \frac{1}{3}t_2\partial_{t_2} + \frac{1}{2}t_3\partial_{t_3} + \frac{7}{12}t_4\partial_{t_4} + \frac{3}{4}t_5\partial_{t_5} + \frac{5}{6}t_6\partial_{t_6} + t_7\partial_{t_7} + \frac{1}{4}t_8\partial_{t_8} \right) \mathbb{F} = \frac{25}{12} \mathbb{F}.$$

In the end we would like to add that we are very impressed by writing this potential in a paper. But we believe that, this potential gives an idea about how complicated algebraic WDVV solutions could be. We write this potential in a Mathematica notebook file and we will upload it in the sub-directory of the arXiv submission. It can also be found on the homepage <http://staffcv.uofk.edu/FMS/Dept-of-Pure-Math/Yassir-Ibrahim/>.

$$\begin{aligned}
F = & \frac{275747251366536586569731516102824113952687921718886400000000t_1^{25}}{26291971297710888812209859761} - \frac{329401897854852815100761902648244065166950400000000\sqrt{\frac{2}{134589}}t_8t_1^{22}}{2736152181450029070729} \\
& - \frac{1945051421153779792882712347780124853351219200000\sqrt{\frac{288230}{3289}}t_2t_1^{21}}{970315921277476588361547} + \frac{60257490748451656689759529090416640000000000\sqrt{\frac{5510}{69069}}Zt_1^{20}}{51511170636379284831} \\
& + \frac{51276965550447936573186450702116293967872000000t_8^2t_1^{19}}{8491244341173937384041} + \frac{8609664200367436587684526517904002252800000\sqrt{\frac{21793}{221}}t_3t_1^{19}}{10845082617828469429419} \\
& + \frac{237507977941170664487849007390455234560000\sqrt{\frac{19}{19499}}t_4t_1^{18}}{380778653527744909437} - \frac{1158908336560292796842981341375391334400000\sqrt{\frac{144115}{2619309}}t_2t_8t_1^{18}}{51327530099351194689} \\
& + \frac{734109387995764004804539557492788341768192000t_2^2t_1^{17}}{6945927852281967485937} - \frac{27453660677782683650021654528000000000\sqrt{\frac{95}{4301}}t_8Zt_1^{17}}{57221326755129537} \\
& - \frac{5452777995210797138612019621629054156800000\sqrt{\frac{2}{134589}}t_8^3t_1^{16}}{2384044104068998371} + \frac{2055990113756671264610881296662528000\sqrt{\frac{190}{383801}}t_5t_1^{16}}{305692228025763579} \\
& + \frac{87138535415522897242687767838720000\sqrt{\frac{1263994}{21}}t_3t_8t_1^{16}}{35723692959645537} + \frac{10638293512640789914383391129600000000\sqrt{\frac{43993}{21}}t_2Zt_1^{16}}{14380803006794963739} \\
& - \frac{26661242077408439262555938492514304000\sqrt{\frac{288230}{3289}}t_2t_8^2t_1^{15}}{240252506611604487} + \frac{237239573650736748213584251337886924800\sqrt{\frac{12710}{4301}}t_2t_3t_1^{15}}{11967241662997207587} \\
& + \frac{552547343072105402364174348478054400\sqrt{\frac{58}{357}}t_6t_1^{15}}{6268555156808061117} + \frac{19572106164686166880069029889835008000\sqrt{\frac{38}{9080799}}t_4t_8t_1^{15}}{26339975807916771} \\
& + \frac{649005721094941689805827190292480\sqrt{\frac{12710}{55913}}t_2t_4t_1^{14}}{3591268931403153} + \frac{9634301289821016796784088857742147584000\sqrt{\frac{2}{134589}}t_2^2t_8t_1^{14}}{367423962740453619} \\
& + \frac{1131978190761833389673676800000000\sqrt{\frac{4370}{87087}}t_8^2Zt_1^{14}}{91039221906633} - \frac{172298337875627126439280640000000\sqrt{\frac{332630}{90321}}t_3Zt_1^{14}}{635305073153751} \\
& + \frac{25908989728497747627499196136816640000t_8^4t_1^{13}}{2309947939218369051} - \frac{27498013460678823586752388578267889664\sqrt{\frac{288230}{3289}}t_2^3t_1^{13}}{22993904272896760203} \\
& + \frac{7756364878391254176136276023758028800t_3^2t_1^{13}}{1407624273750994299} - \frac{69171011505030796144541696000\sqrt{\frac{283309}{17}}t_3t_8^2t_1^{13}}{1889703018819} \\
& + \frac{37858410608405201801558622208000\sqrt{\frac{2185}{2245886643}}t_5t_8t_1^{13}}{3178802752587} - \frac{4753057596569024177635328000000\sqrt{\frac{10}{39056516499}}t_4Zt_1^{13}}{59167157049} \\
& + \frac{12117169859674321505484800000000\sqrt{\frac{3034}{221}}t_2t_8Zt_1^{13}}{175548954964383} + \frac{682332614278050128408269448806400\sqrt{\frac{3314645}{113883}}t_2t_8^3t_1^{12}}{2630725180621083}
\end{aligned}$$

$$\begin{aligned}
& -\frac{581596580070975576125721070796800\sqrt{\frac{19}{19499}}t_4t_8^2t_1^{12}}{2168478895654071} - \frac{59277167362036445126003261440t_3t_4t_1^{12}}{70589452572639\sqrt{13}} + \frac{56181243169575977303136010240t_2t_5t_1^{12}}{23529817524213\sqrt{13}} \\
& -\frac{245825012333495379886125130711040\sqrt{\frac{184295}{69069}}t_2t_3t_8t_1^{12}}{275940587329407} + \frac{236615066302532511259741388800t_6t_8t_1^{12}}{23529817524213\sqrt{13}} + \frac{3748908861782265246305484800000\sqrt{\frac{5510}{69069}}t_2^2t_1^{12}}{93898278236763} \\
& + \frac{3775956970890476910732497649664t_4^2t_1^{11}}{906846213331879821} + \frac{5002923413852734025203121190338560t_2^2t_8t_1^{11}}{28745698069975887} + \frac{35514283134452206559049209085952\sqrt{\frac{21793}{221}}t_2^2t_3t_1^{11}}{2196577671230943} \\
& + \frac{24634735902975255837197467648\sqrt{\frac{835867}{5870865}}t_2t_6t_1^{11}}{59121138149073} - \frac{48882242654346106005782265856\sqrt{\frac{6355}{154077}}t_2t_4t_8t_1^{11}}{9688472554761} - \frac{62651926800793631457280000000\sqrt{\frac{2185}{187}}t_8^3Zt_1^{11}}{67454491810797} \\
& - \frac{411448891669756248064000000\sqrt{\frac{29}{414141}}t_5Zt_1^{11}}{348506614071} + \frac{78500117489624547328000000\sqrt{\frac{5735}{3289}}t_3t_8Zt_1^{11}}{54286954029} - \frac{30579366843514835600438422667264000\sqrt{\frac{2}{134589}}t_8^5t_1^{10}}{10304778647825487} \\
& + \frac{22232868668243672938263347200\sqrt{\frac{43586}{609}}t_3t_8^3t_1^{10}}{17236382080743} - \frac{125348126861275384690769920\sqrt{\frac{161690}{451}}t_5t_8^2t_1^{10}}{40040071035111} - \frac{189760322042691934295687168\sqrt{370481}t_2^2t_4t_1^{10}}{936498700932795} \\
& - \frac{14880867845386572962725888\sqrt{\frac{130}{5466571}}t_3t_5t_1^{10}}{40539870063} - \frac{62830748616325803253808902438912\sqrt{\frac{2449955}{154077}}t_2^3t_8t_1^{10}}{20677539028683537} - \frac{95011219805712869211813969920\sqrt{\frac{754}{357}}t_3^2t_8t_1^{10}}{40093931492307} \\
& - \frac{161203054336679084032000000\sqrt{\frac{1517}{609}}t_2t_8^2Zt_1^{10}}{1105051246299} + \frac{29618300908091893350400000\sqrt{\frac{700321}{4641}}t_2t_3Zt_1^{10}}{13643339572863} - \frac{33755050520538555351040000\sqrt{\frac{95}{55913}}t_6Zt_1^{10}}{593188677081} \\
& + \frac{2165520482472401305600000\sqrt{\frac{115}{12617}}t_4t_8Zt_1^{10}}{106466731371} + \frac{379599576880637346221842511367847936t_2^4t_1^9}{248022271919052355725} - \frac{31529410268465987578882949120\sqrt{\frac{72922190}{13}}t_2t_8^4t_1^9}{28038583762687953} \\
& + \frac{197407620359075269146711162880\sqrt{\frac{38}{9080799}}t_4t_8^3t_1^9}{31050480357351} - \frac{3160121833455978507411980288\sqrt{\frac{57646}{16445}}t_2t_3^2t_1^9}{7148530421109} + \frac{217861791934905227836129280\sqrt{\frac{12710}{4301}}t_2t_3t_8^2t_1^9}{306529366689} \\
& - \frac{1260293666811736204771328\sqrt{\frac{46}{10353}}t_6t_8^2t_1^9}{1244674431} - \frac{2052533495915389374169088\sqrt{\frac{46}{1188385}}t_4t_5t_1^9}{2888484111513} - \frac{1052429798275695127298048\sqrt{\frac{1263994}{273}}t_3t_6t_1^9}{168501074211855} \\
& - \frac{109423644938076252274688\sqrt{\frac{2}{10353}}t_3t_4t_8t_1^9}{86837751} - \frac{185359867371260393553920\sqrt{\frac{2}{10353}}t_2t_5t_8t_1^9}{339456663} - \frac{712342263971184640000\sqrt{\frac{410533}{609}}t_2t_4Zt_1^9}{280809205677} \\
& + \frac{34160490430294733619200000\sqrt{\frac{1615}{253}}t_2^2t_8Zt_1^9}{23051882975121} - \frac{54459877768149145360567107584\sqrt{\frac{34}{7917}}t_2^2t_3^2t_1^8}{13919180849847} + \frac{169398250083998769348608\sqrt{\frac{993922}{715}}t_2t_4t_8^2t_1^8}{3129118175547} \\
& + \frac{4431201496798079746048\sqrt{\frac{57646}{1265}}t_2t_3t_4t_1^8}{22909841955} - \frac{5205387859279057518592\sqrt{\frac{57646}{1265}}t_2^2t_5t_1^8}{129822437745} - \frac{145162730796647603765248\sqrt{\frac{38}{698523}}t_4t_6t_1^8}{1020033439425} \\
& - \frac{11548884065344094489018368\sqrt{\frac{34}{7917}}t_4^2t_8t_1^8}{218605116394665} + \frac{105526751443021144048369664\sqrt{\frac{1263994}{21}}t_2^2t_3t_8t_1^8}{128394390929805} - \frac{1890182858062194802688\sqrt{\frac{57646}{1265}}t_2t_6t_8t_1^8}{7081223877}
\end{aligned}$$

$$\begin{aligned}
& + \frac{2391235564049268736000000\sqrt{\frac{48070}{7917}}t_8^4Zt_1^8}{1180522241703} + \frac{22503223070956655771648000\sqrt{\frac{43993}{21}}t_2^3Zt_1^8}{1931124907440729} + \frac{7435342160099246080000\sqrt{\frac{71630}{5313}}t_3^2Zt_1^8}{133202430207} \\
& - \frac{28365429486387200000\sqrt{\frac{7650490}{3927}}t_3t_8^2Zt_1^8}{374272731} + \frac{10778863204827136000000\sqrt{\frac{2}{25789}}t_5t_8Zt_1^8}{8904209391} + \frac{21434549150503361944170895769600t_8^6t_1^7}{4283352991279460763} \\
& - \frac{4287547480979049202319360\sqrt{\frac{283309}{17}}t_3t_8^4t_1^7}{46827382254219} + \frac{469684633100594953600000\sqrt{\frac{283309}{17}}t_3^3t_1^7}{21445591263327} + \frac{74052171417443891150848\sqrt{\frac{11560835}{424473}}t_5t_8^3t_1^7}{4822753654089} \\
& - \frac{265329980759757291520\sqrt{\frac{7790}{121693}}t_2t_4^2t_1^7}{103715653041} + \frac{2828941945374762065199104t_5^2t_1^7}{10944541213524765} - \frac{15887825476612819786628661248\sqrt{\frac{57646}{16445}}t_2^3t_8^2t_1^7}{1451802932827065} \\
& + \frac{5706043389286903615127552t_3^2t_8^2t_1^7}{757153734813} - \frac{381865658996990733088897719296\sqrt{\frac{2542}{21505}}t_2^3t_3t_1^7}{7089794240604075} - \frac{328941634965862444714172416\sqrt{\frac{58}{357}}t_2^2t_6t_1^7}{2063167636154625} \\
& - \frac{1287399644481804608602112\sqrt{\frac{43586}{7917}}t_2^2t_4t_8t_1^7}{88426535222025} - \frac{1875868881215533088768\sqrt{\frac{667}{1468005}}t_3t_5t_8t_1^7}{838486971} + \frac{18234492407722803200000\sqrt{\frac{51578}{13}}t_2t_8^3Zt_1^7}{1070706219219} \\
& - \frac{90769374356439040000\sqrt{\frac{43010}{9080799}}t_4t_8^2Zt_1^7}{1185721173} + \frac{12332795428864000\sqrt{\frac{5510}{5313}}t_3t_4Zt_1^7}{12405393} - \frac{6166397714432000\sqrt{\frac{5510}{5313}}t_2t_5Zt_1^7}{12405393} - \frac{256391108968939520000\sqrt{48298}t_2t_3t_8Zt_1^7}{257647607217} \\
& + \frac{30831988572160000\sqrt{\frac{5510}{5313}}t_6t_8Zt_1^7}{12405393} + \frac{46404673915505485286998016\sqrt{\frac{32623085}{11571}}t_2t_8^5t_1^6}{2126382895583037} - \frac{64219603583178549716058112\sqrt{\frac{323}{1147}}t_4t_8^4t_1^6}{4302216556179633} \\
& - \frac{1871220704220541288448\sqrt{\frac{321563}{39585}}t_2t_3t_8^3t_1^6}{2299103919} + \frac{3722731443539978223616t_6t_8^3t_1^6}{54269545995\sqrt{13}} + \frac{698055308833678426112t_3t_4t_8^2t_1^6}{18089848665\sqrt{13}} + \frac{106374023157810987008t_2t_5t_8^2t_1^6}{2584264095\sqrt{13}} \\
& + \frac{296437133234144764850176\sqrt{\frac{216070}{3289}}t_2^3t_4t_1^6}{221354238402297} + \frac{183825589760000\sqrt{\frac{21793}{17}}t_3^2t_4t_1^6}{51212007} - \frac{16767557595947008\sqrt{\frac{21793}{17}}t_2t_3t_5t_1^6}{55665225} - \frac{12566025865360769024\sqrt{\frac{551}{685084785}}t_5t_6t_1^6}{1396932075} \\
& + \frac{4508542535356502346462936064\sqrt{\frac{34}{7917}}t_2^4t_8t_1^6}{686267529090507} - \frac{3056148141650068087808\sqrt{\frac{835867}{451605}}t_2t_3^2t_8t_1^6}{5672117745} - \frac{6213465218395906506752\sqrt{\frac{23}{553437885}}t_4t_5t_8t_1^6}{22977914085} \\
& - \frac{6882616087455776768\sqrt{\frac{21793}{17}}t_3t_6t_8t_1^6}{21765102975} + \frac{83863008916275200\sqrt{\frac{4370}{87087}}t_4^2Zt_1^6}{15026127831} + \frac{16457073999942434816000\sqrt{\frac{190}{2003001}}t_2^2t_8^2Zt_1^6}{1594590543} - \frac{42878039407831859200\sqrt{\frac{332630}{90321}}t_2^2t_3Zt_1^6}{15055038999} \\
& - \frac{1489840227792486400\sqrt{\frac{3034}{17}}t_2t_6Zt_1^6}{165605428989} - \frac{104828761145344000\sqrt{\frac{48298}{13}}t_2t_4t_8Zt_1^6}{11829636819} - \frac{1540964400458670051556291472\sqrt{\frac{57646}{16445}}t_2^5t_1^5}{52491486120434361} \\
& + \frac{2950962167502149088348848128t_2^2t_8^4t_1^5}{26173583482176585} - \frac{397577030236771188736\sqrt{\frac{321563}{3045}}t_2t_4t_8^3t_1^5}{3559578876945} + \frac{23921384687434934326958624t_2^2t_3^2t_1^5}{172706092106625} + \frac{96349964288000\sqrt{\frac{323}{14911}}t_3t_4^2t_1^5}{2351349} \\
& + \frac{1607790932517785666048t_6^2t_1^5}{255662227445625} + \frac{32316116749389439434752t_4^2t_8^2t_1^5}{5900445457665525} - \frac{34013318595020812490752\sqrt{\frac{1147}{4199}}t_2^2t_3t_8^2t_1^5}{164569921425}
\end{aligned}$$

$$\begin{aligned}
& + \frac{235811979254300672\sqrt{\frac{28823}{170255085}}t_2t_6t_8^2t_1^5}{1131975} + \frac{1953069824573440\sqrt{\frac{19}{253487}}t_2t_4t_5t_1^5}{2351349} - \frac{30272806725074369536\sqrt{\frac{36859}{26565}}t_2t_3t_6t_1^5}{39062219625} + \frac{785230641485037568\sqrt{\frac{28823}{170255085}}t_2t_3t_4t_8t_1^5}{7923825} \\
& + \frac{2898792624381952\sqrt{\frac{144115}{34051017}}t_2^2t_5t_8t_1^5}{316953} - \frac{250359054902755328\sqrt{\frac{19}{253487}}t_4t_6t_8t_1^5}{293918625} - \frac{181117057787494400000\sqrt{408595}t_8^5Zt_1^5}{54522638348283} + \frac{6616808161280000\sqrt{18862415}t_3t_8^3Zt_1^5}{5364575811} \\
& - \frac{587345111941120000\sqrt{\frac{13}{923853}}t_5t_8^2Zt_1^5}{61388649} - \frac{8765615487877120\sqrt{\frac{194990}{2003001}}t_2^2t_4Zt_1^5}{135069363} + \frac{402481291264000\sqrt{\frac{551}{453747}}t_3t_5Zt_1^5}{1802493} + \frac{205051967041083392000\sqrt{\frac{51578}{13}}t_2^3t_8Zt_1^5}{2805255600147} \\
& - \frac{352308820824064000\sqrt{\frac{95}{4301}}t_3^2t_8Zt_1^5}{49322763} - \frac{2158936234052031567626240000\sqrt{\frac{442}{609}}t_8^7t_1^4}{2645124258409160577} + \frac{8695021482453436416000\sqrt{\frac{6882}{3857}}t_3t_8^5t_1^4}{3551282957551} - \frac{336957034612426342400\sqrt{\frac{48070}{1517}}t_5t_8^4t_1^4}{74246590205543} \\
& + \frac{26965819478835200\sqrt{\frac{323}{1147}}t_4^3t_1^4}{54760218525441} + \frac{932722984197401182208\sqrt{\frac{26949505}{14007}}t_2^3t_8^3t_1^4}{8616635764191} - \frac{25629396450036121600\sqrt{\frac{754}{357}}t_2^3t_8^3t_1^4}{10133087643} - \frac{5833918962368512\sqrt{370481}t_2^2t_4t_8^2t_1^4}{45048212937} \\
& + \frac{10976384438456320\sqrt{\frac{32890}{21607}}t_3t_5t_8^2t_1^4}{235653201} - \frac{13241958273039104t_2^2t_3t_4t_1^4}{21724443\sqrt{13}} - \frac{34145463252284672t_2^3t_5t_1^4}{386026641\sqrt{13}} + \frac{31195609952000\sqrt{\frac{7030}{10373}}t_3^2t_5t_1^4}{928557} - \frac{44167651011584\sqrt{\frac{8897}{1430715}}t_2t_4t_6t_1^4}{3463317} \\
& - \frac{37664594331520000\sqrt{\frac{1263994}{21}}t_3^3t_8t_1^4}{19285200333} - \frac{16549999230976\sqrt{\frac{1522945}{247863}}t_2^2t_4t_8t_1^4}{238350333} + \frac{442283885249822720\sqrt{\frac{26}{10353}}t_5^2t_8t_1^4}{380447348379} - \frac{2792998290203006336\sqrt{\frac{405449}{31395}}t_2^3t_8t_1^4}{726561927} \\
& + \frac{144777055201009664t_2^3t_6t_8t_1^4}{363648285\sqrt{13}} + \frac{269622764011015604416\sqrt{\frac{5510}{69069}}t_4^2Zt_1^4}{928872009351} - \frac{61124597799485440000\sqrt{\frac{1517}{609}}t_2t_8^4Zt_1^4}{243599827653} + \frac{493179660795904000\sqrt{\frac{1265}{1147}}t_4t_8^3Zt_1^4}{164288255859} \\
& - \frac{2625378232102400\sqrt{\frac{43993}{21}}t_2t_3^2Zt_1^4}{2860556391} + \frac{23320253440000\sqrt{\frac{700321}{4641}}t_2t_3t_8^2Zt_1^4}{349461} - \frac{5618892800\sqrt{\frac{28405}{187}}t_6t_8^2Zt_1^4}{8127} + \frac{19643649228800\sqrt{\frac{19}{171062619}}t_4t_5Zt_1^4}{340659} \\
& + \frac{48259054827520\sqrt{\frac{5735}{253}}t_3t_6Zt_1^4}{124372017} - \frac{11237785600\sqrt{\frac{2185}{2431}}t_3t_4t_8Zt_1^4}{3483} - \frac{112377856000\sqrt{\frac{2185}{2431}}t_2t_5t_8Zt_1^4}{24381} - \frac{129703677227242075468800\sqrt{\frac{3838010}{247}}t_2t_8^6t_1^3}{1291804870385869119} \\
& + \frac{109310473017890278604800\sqrt{\frac{6}{57511727}}t_4t_8^5t_1^3}{25097722414289} + \frac{1008644456653419520\sqrt{\frac{3215630}{17}}t_2t_3t_8^4t_1^3}{830737873791} - \frac{78331776737280\sqrt{\frac{6}{3451}}t_6t_8^4t_1^3}{556549} - \frac{323750049601694950\sqrt{\frac{12710}{4301}}t_2t_3^3t_1^3}{1820354067} \\
& - \frac{4905667695554560\sqrt{\frac{2}{10353}}t_3t_4t_8^3t_1^3}{16139921} - \frac{11919985887293440\sqrt{\frac{2}{10353}}t_2t_5t_8^3t_1^3}{48419763} + \frac{33332406263111680t_2^2t_4^3t_1^3}{492237781893} - \frac{11695614464\sqrt{\frac{56810}{16687}}t_2^2t_5^3t_1^3}{564417} + \frac{324321642022604931470608t_2^4t_8^2t_1^3}{1378060561635849} \\
& - \frac{4735581833401184\sqrt{\frac{2169310}{437}}t_2t_3^2t_8^2t_1^3}{706959603} + \frac{457125753110528\sqrt{\frac{2530}{21607}}t_4t_5t_8^2t_1^3}{5350785969} + \frac{10308305792\sqrt{\frac{16431922}{21}}t_3t_6t_8^2t_1^3}{782901} + \frac{3337040790377704912907\sqrt{\frac{370481}{13}}t_2^4t_3t_1^3}{1035335031961230} \\
& - \frac{702361600\sqrt{\frac{56810}{16687}}t_3t_4t_5t_1^3}{33201} + \frac{35772423689400752\sqrt{\frac{99468173}{49335}}t_2^3t_6t_1^3}{1071775395405} - \frac{4808120870133100\sqrt{\frac{58}{357}}t_3^2t_6t_1^3}{317839599} + \frac{10524737455707266816\sqrt{\frac{1271}{770385}}t_2^3t_4t_8t_1^3}{199055403963} \\
& - \frac{11552000\sqrt{\frac{16431922}{21}}t_3^2t_4t_8t_1^3}{2709} + \frac{33162496\sqrt{\frac{16431922}{21}}t_2t_3t_5t_8t_1^3}{111843} - \frac{92015148032\sqrt{\frac{11362}{83435}}t_5t_6t_8t_1^3}{564417} - \frac{2989531521228800\sqrt{408595}t_2^2t_8^3Zt_1^3}{73646459523}
\end{aligned}$$

$$\begin{aligned}
& + \frac{231261798400\sqrt{\frac{410533}{609}}t_2t_4t_8^2Zt_1^3}{18820971} - \frac{2177320960\sqrt{\frac{43993}{273}}t_2t_3t_4Zt_1^3}{266409} - \frac{1088660480\sqrt{\frac{43993}{273}}t_2^2t_5Zt_1^3}{88803} + \frac{45906354176\sqrt{\frac{5}{3772483}}t_4t_6Zt_1^3}{11583} \\
& - \frac{3515179335680\sqrt{\frac{37145}{11}}t_4^2t_8Zt_1^3}{189268499043} + \frac{742348327697920\sqrt{\frac{5735}{3289}}t_2^2t_3t_8Zt_1^3}{21869757} - \frac{21773209600\sqrt{\frac{43993}{273}}t_2t_6t_8Zt_1^3}{266409} - \frac{155540097158692339850240\sqrt{\frac{102}{2639}}t_2^2t_8^2t_1^2}{4061503837595113} \\
& + \frac{11540725369823232\sqrt{\frac{54665710}{13}}t_2t_4t_8^4t_1^2}{144165986891381} - \frac{405070505245294592\sqrt{\frac{34}{7917}}t_4^2t_8^3t_1^2}{28913863781153} + \frac{90733129640594848\sqrt{\frac{2294}{11571}}t_2^2t_3t_8^3t_1^2}{1237208427} - \frac{360361606016\sqrt{\frac{767602}{95}}t_2t_6t_8^3t_1^2}{43915599} \\
& - \frac{10159893376\sqrt{\frac{767602}{95}}t_2t_3t_4t_8^2t_1^2}{6273657} - \frac{710272\sqrt{72922190}t_2^2t_5t_8^2t_1^2}{48633} + \frac{32689198592\sqrt{\frac{266}{99789}}t_4t_6t_8^2t_1^2}{243165} - \frac{164315483619570550\sqrt{370481}t_2^4t_4t_1^2}{350184747525327} + \frac{240158860}{161}\sqrt{\frac{12710}{55913}}t_2^2t_3t_4t_1^2 \\
& + \frac{54924677120\sqrt{\frac{4370}{16687}}t_2^2t_5t_1^2}{1166234043} + \frac{35601655004\sqrt{\frac{33046}{21505}}t_2^2t_3t_5t_1^2}{73899} - \frac{47189920\sqrt{\frac{58}{4641}}t_3t_4t_6t_1^2}{2277} + \frac{4668324928\sqrt{\frac{58}{4641}}t_2t_5t_6t_1^2}{193545} - \frac{153751727659501843325\sqrt{\frac{2449955}{154077}}t_2^5t_8t_1^2}{94407173314038} \\
& + \frac{928913367501436\sqrt{\frac{406}{663}}t_2^2t_3t_8t_1^2}{58346055} - \frac{13129464512\sqrt{\frac{406}{663}}t_6^2t_8t_1^2}{967725} + \frac{35542629376\sqrt{\frac{38}{698523}}t_2t_4t_5t_8t_1^2}{48633} + \frac{3915730205848\sqrt{\frac{2542}{279565}}t_2t_3t_6t_8t_1^2}{369495} \\
& + \frac{5705857874083840000\sqrt{\frac{32890}{11571}}t_8^6Zt_1^2}{37213864269463} - \frac{1704175616000\sqrt{\frac{2901910}{10353}}t_3t_8^4Zt_1^2}{23931607} + \frac{1489449488384000\sqrt{\frac{2}{25789}}t_5t_8^3Zt_1^2}{1501012653} + \frac{1105295360\sqrt{\frac{1647490}{231}}t_5^2Zt_1^2}{1263944679} \\
& - \frac{317637724130560\sqrt{\frac{1517}{609}}t_2^3t_8^2Zt_1^2}{840886371} + \frac{5534963200\sqrt{\frac{18122390}{21}}t_3^2t_8^2Zt_1^2}{3740527} + \frac{7370788883912\sqrt{\frac{11905457}{273}}t_2^3t_3Zt_1^2}{1832094693} - \frac{21000643480928\sqrt{\frac{323}{16445}}t_2^2t_6Zt_1^2}{79656291} \\
& + \frac{135784610816\sqrt{\frac{5735}{253}}t_2^2t_4t_8Zt_1^2}{73970793} - \frac{8435148800\sqrt{\frac{494}{1271}}t_3t_5t_8Zt_1^2}{260967} + \frac{1453995723110575582080000000t_8^8t_1}{146232167793805896952009} + \frac{4916728097717507366510447t_2^6t_1}{21741928103909782080} \\
& - \frac{3098936846720000000\sqrt{\frac{253487}{19}}t_3t_8^6t_1}{4949443947838579} + \frac{40567900538880000\sqrt{\frac{838695}{5851069}}t_5t_8^5t_1}{26828599822171} + \frac{255540004042051625t_3^4t_1}{534432838464} - \frac{2248830784740736064\sqrt{\frac{3838010}{247}}t_2^3t_8^4t_1}{1744905565478439} \\
& + \frac{11166926928300000t_3^2t_8^4t_1}{54109363427} + \frac{1154280198676352\sqrt{\frac{2294}{150423}}t_2^2t_4t_8^3t_1}{33900751209} - \frac{314691520000\sqrt{\frac{3795}{258013}}t_3t_5t_8^3t_1}{9461333} - \frac{370421080518379193\sqrt{\frac{1517}{624910}}t_2^3t_3^2t_1}{7623051345} \\
& - \frac{392768\sqrt{\frac{216070}{253}}t_2t_3t_4^2t_1}{44109} + \frac{4283474000\sqrt{\frac{247}{19499}}t_3t_5^2t_1}{2571233} - \frac{468277715963\sqrt{\frac{28823}{32890}}t_2t_6^2t_1}{868049325} + \frac{1}{2}t_7^2t_1 + \frac{1120775995625\sqrt{\frac{14911}{323}}t_3^3t_8^2t_1}{78551067} - \frac{105422908928\sqrt{\frac{394174}{2405}}t_2t_4^2t_8^2t_1}{22863297327} \\
& + \frac{412560582720000t_5^2t_8^2t_1}{9302310655463} + \frac{135254310447593\sqrt{\frac{475354}{115}}t_2^3t_3t_8^2t_1}{25808743233} - \frac{1043811292\sqrt{\frac{238}{87}}t_2^2t_6t_8^2t_1}{58305} - \frac{490960\sqrt{\frac{216070}{253}}t_2^2t_4t_5t_1}{44109} - \frac{320891456\sqrt{\frac{238}{87}}t_4^2t_6t_1}{185507751} \\
& - \frac{19489949684819\sqrt{\frac{631997}{546}}t_2^2t_3t_6t_1}{79860537900} - \frac{19214964674560\sqrt{\frac{266}{1297257}}t_4^3t_8t_1}{609865163583} - \frac{82122873884\sqrt{\frac{34}{609}}t_2^2t_3t_4t_8t_1}{578565} - \frac{47088135100\sqrt{\frac{34}{609}}t_2^3t_5t_8t_1}{1504269} + \frac{4901000\sqrt{\frac{335264215}{14637}}t_3^2t_5t_8t_1}{86989} \\
& + \frac{18834016\sqrt{\frac{43214}{1265}}t_2t_4t_6t_8t_1}{220545} + \frac{8739209194496000\sqrt{\frac{51578}{13}}t_2t_8^5Zt_1}{16443335374879} - \frac{20502543564800\sqrt{\frac{43010}{9080799}}t_4^4t_8Zt_1}{2875211641} - \frac{53859449600\sqrt{\frac{2542}{19}}t_2t_3t_8^3Zt_1}{5008941} + \frac{1295360\sqrt{\frac{48070}{609}}t_6t_8^3Zt_1}{5547}
\end{aligned}$$

$$\begin{aligned}
& + \frac{235520\sqrt{\frac{48070}{609}t_3t_4t_8^2Zt_1}}{1849} + \frac{117760\sqrt{\frac{48070}{609}t_2t_5t_8^2Zt_1}}{1849} - \frac{17632}{153}\sqrt{\frac{4324190}{90321}t_2t_3t_5Zt_1} + \frac{57621376\sqrt{\frac{26}{25789}t_5t_6Zt_1}}{52173} + \frac{2392274960\sqrt{\frac{39442}{17}t_2t_3^2t_8Zt_1}}{1059219} \\
& - \frac{205844480\sqrt{\frac{38}{1271}t_4t_5t_8Zt_1}}{4290927} - \frac{35264}{51}\sqrt{\frac{4324190}{90321}t_3t_6t_8Zt_1} + \frac{93968346863818240000\sqrt{\frac{97869255}{3857}t_2t_8^7}}{16752453636591350321} - \frac{380282459448883200000\sqrt{\frac{17}{21793}t_4t_8^6}}{594640337161749277} \\
& - \frac{109833811208000\sqrt{\frac{62704785}{203}t_2t_3t_8^5}}{10773219728369} + \frac{1441585094400\sqrt{13t_6t_8^5}}{54629021179} + \frac{9610567296000\sqrt{13t_3t_4t_8^4}}{382403148253} + \frac{2402641824000\sqrt{13t_2t_5t_8^4}}{382403148253} + \frac{794569664\sqrt{\frac{32062}{22165}t_2t_4^3}}{17843016789} \\
& + \frac{574284891088122335\sqrt{\frac{34}{7917}t_2^4t_8^3}}{59769362364309} + \frac{1398279025\sqrt{\frac{55651145}{6783}t_2t_3^2t_8^3}}{10574431} - \frac{174737587200\sqrt{\frac{49335}{258013}t_4t_5t_8^3}}{76694598889} - \frac{3946800\sqrt{\frac{1147}{323}t_3t_6t_8^3}}{12943} - \frac{125436480\sqrt{\frac{19}{19499}t_4t_8^2}}{42277273} \\
& + \frac{28649412724\sqrt{\frac{2376770}{299}t_2^3t_4t_8^2}}{30747192957} - \frac{328900\sqrt{\frac{1147}{323}t_3^2t_4t_8^2}}{12943} - \frac{1973400\sqrt{\frac{1147}{323}t_2t_3t_5t_8^2}}{12943} + \frac{95680\sqrt{\frac{72105}{5235167}t_5t_6t_8^2}}{1333} - \frac{62700755\sqrt{13t_3^3t_4}}{1675044} + \frac{13060761683\sqrt{\frac{28823}{2530}t_2^3t_3t_4}}{41839668} \\
& + \frac{13060761683\sqrt{\frac{28823}{2530}t_2^4t_5}}{251038008} - \frac{62700755\sqrt{13t_2t_3^2t_5}}{186116} + \frac{300099623\sqrt{\frac{152551}{174}t_2^2t_4t_6}}{575295435} - \frac{308009}{289}\sqrt{\frac{1885}{6752823}t_3t_5t_6 + t_3t_4t_7 + t_2t_5t_7} - \frac{791095425835\sqrt{\frac{26354185}{483}t_2t_3^2t_8}}{5474043792} \\
& + \frac{1898304592\sqrt{\frac{238}{1131}t_2^2t_4t_8}}{238350333} - \frac{23920\sqrt{\frac{72105}{5235167}t_2t_5t_8}}{1333} + \frac{614208439666441\sqrt{\frac{232841}{114}t_2^4t_3t_8}}{993106359648} - \frac{47840\sqrt{\frac{72105}{5235167}t_3t_4t_5t_8}}{1333} - \frac{13060761683\sqrt{\frac{28823}{2530}t_2^3t_6t_8}}{125519004} + \frac{62700755\sqrt{13t_3^2t_6t_8}}{279174} \\
& + t_6t_7t_8 + \frac{68219307474560\sqrt{\frac{7590}{50141}t_2^4t_8^4Z}}{24488871563} - \frac{8610257920\sqrt{\frac{2542}{247}t_2t_4t_8^3Z}}{869250031} + \frac{1406268931799\sqrt{\frac{29}{13123110}t_2^2t_3^2Z}}{971244} + \frac{2037598\sqrt{\frac{7714}{49335}t_6^2Z}}{193545} + \frac{313100288\sqrt{\frac{336490}{1131}t_4^2t_8^2Z}}{23186739199} \\
& - \frac{11662696\sqrt{\frac{62201810}{483}t_2^2t_3t_8^2Z}}{1370109} + \frac{992\sqrt{51578t_2t_6t_8^2Z}}{1677} - \frac{41478961\sqrt{\frac{48298}{13}t_2t_3t_6Z}}{13354605} + \frac{992\sqrt{51578t_2t_3t_4t_8Z}}{5031} - \frac{2432}{39}\sqrt{\frac{5474}{5488395}t_4t_6t_8Z}
\end{aligned}$$

Where Z is a solution of the quadratic equation

$$\begin{aligned}
0 = & Z^2(6087380819159303391005603325000\sqrt{1168090}) + Z(199260374254874324273405781024768000\sqrt{749772699}t_1^5 \\
& - 1505637451640394074610851443200000\sqrt{157777102}t_8t_1^2 - 7430529091408703313576581198160000\sqrt{9570}t_2t_1) \\
& + 326963614070713431645521199551350841540608\sqrt{1168090}t_1^{10} - 174838428297999356567466344499354009600\sqrt{1604204745}t_8t_1^7 \\
& - 1711334762076397518024060769393648435200\sqrt{39767}t_2t_1^6 + 9960759444454760229501745087395840000\sqrt{1168090}t_8^2t_1^4 \\
& + 9556665279777336201987345840000000\sqrt{4109435330}t_3t_1^4 + 85101765451358857061724864000000\sqrt{316110410}t_4t_1^3 \\
& + 616317099570460802320908222401280000\sqrt{1292646}t_2t_8t_1^3 + 7240507609750674488274075585786000\sqrt{1168090}t_2^2t_1^2 \\
& + 17177727076618842365551920000000\sqrt{1604204745}t_8^3t_1 + 210257920682101729710627354000000\sqrt{3059}t_5t_1 \\
& - 147592310295208496926615407000000\sqrt{115552965}t_3t_8t_1 + 223400586725581166350207258500000\sqrt{39767}t_2t_8^2 \\
& - 35432800382670021674509694653125\sqrt{3139339}t_2t_3 + 248545233845734676184486652500\sqrt{123400365}t_6 - 16850054997131291207672400000\sqrt{1502188545}t_4t_8
\end{aligned}$$

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